ON SEMIGROUPS GENERATED BY RESTRICTIONS OF ELLIPTIC OPERATORS TO INVARIANT SUBSPACES

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ABSTRACT

Let A be the closed unbounded operator in $L_p(G)$ that is associated with an elliptic boundary value problem for a bounded domain G. We prove the existence of a spectral projection E determined by the set $\Gamma = \{\lambda; \theta_1 \leqq \arg \lambda \leqq \theta_2\}$ and show that AE is the infinitesimal generator of an analytic semigroup provided that the following conditions hold: $1 < p < \infty$; the boundary $\partial\Gamma$ of Γ is contained in the resolvent set $\rho(A)$ of A; $\pi/2 \leqq \theta_1 < \theta_2 \leqq 3\pi/2$; and there exists a constant c such that (I) $\| (\lambda - A)^{-1} \| \leqq c/|\lambda|$ for $\lambda \in \partial\Gamma$. The following consequence is obtained: Suppose that there exist constants M and c such that $\lambda \in \rho(A)$ and estimate (I) holds provided that $|\lambda| \geqq M$ and $\operatorname{Re} \lambda = 0$. Then there exist bounded projections E^- and E^+ such that A is completely reduced by the direct sum decomposition $L_p(G) = E^- L_p(G) \oplus E^+ L_p(G)$ and each of the operators AE^- and $-AE^+$ is the infinitesimal generator of an analytic semigroup.

1. Introduction

Let A be the unbounded operator in $L_p(G)$ that is associated with an elliptic boundary value problem for a bounded domain G in R^ν. Suppose that for $j = 1,2$, $l_{\alpha_j} = \{\lambda; \lambda = re^{i\alpha_j}, r \geqq 0\}$ is a ray of minimal growth of the resolvent of A.

Recall that l_θ is a ray of minimal growth of the resolvent of A if there exist constants M and c such that $\lambda \in \rho(A)$ and

$$\|(\lambda - A)^{-1}\| \leqq c/|\lambda| \tag{1.1}$$

provided that $\lambda \in l_\theta$ and $|\lambda| \geqq M$. Here $\rho(A)$ is the resolvent set of A.

Assume in addition that for $j = 1,2$, $l_{\alpha_j} \subset \rho(A)$ and that $0 < \alpha_2 - \alpha_1 \leqq \pi$. Suppose that $1 < p < \infty$. We prove the existence of a "spectral" projection E of A that corresponds to the set $\Gamma(\alpha_1, \alpha_2) = \{\lambda; \alpha_1 \leqq \arg \lambda \leqq \alpha_2\}$ and show that

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the operator AE satisfies the following condition: There exists a constant c such that the estimate

$$\| (\lambda - AE)^{-1} \| \leqq c / |\lambda| \tag{1.2}$$

is satisfied for $\lambda \notin \Gamma(\alpha_1, \alpha_2)$.

The following consequence is obtained. Suppose that $l_{\pi/2}$ and $l_{-\pi/2}$ are rays of minimal growth of the resolvent of A. Then there exist bounded projections E^- and E^+ in $L_p(G)$ such that the following assertions are satisfied: $L_p(G) = E^- L_p(G) \oplus E^+ L_p(G)$ and A is completely reduced by this direct sum decomposition. Moreover both AE^- and $-AE^+$ are infinitesimal generators of analytic semigroups.

These results improve the results of [2] where it was assumed that $p = 2$, that the elliptic system associated with A reduces to a single differential operator of order ω and that there exist rays of minimal growth of the resolvent of A that divide the complex plane into angles less than $\omega\pi/v$. The last assumption implies that the generalized eigenfunctions of A are complete in $L_p(G)$. See [1]. This condition, however, is superfluous for various applications, for example, for the study of the existence and regularity of solutions of the parabolic equation $du/dt - A(t)u = f$. See [3].

The author is indebted to Professor R.T. Seeley for the communication of the results of [6] prior to publication and for suggesting that Lemma 3 might be useful in extending the results of [2] to systems.

The notations and definitions that we use are introduced in Section 2. Section 3 contains a preliminary result and a description of those results of [1], [5] and [6] that we use in Section 1. The main results are stated and proved in Section 4.

2. Notations and definitions

Let G be a bounded domain in R^v with an infinitely differentiable boundary ∂G and closure $\bar{G}$. Denote by $C^\infty(G)$ $(C^\infty(\bar{G}))$ the set of l-tuples of infinitely differentiable complex valued functions that are defined in G $(\bar{G})$. As usual $C_0^\infty(G)$ is the subset of $C^\infty(G)$ consisting of those functions in $C^\infty(G)$ the support of which is a compact subset of G. For $1 < p < \infty$ and $\omega = 0, 1, \cdots, H^{\omega,p}(G)$ is the completion of $C^\infty(\bar{G})$ under the norm

$$\sum_{|\alpha| \leqq \omega} \left(\int_G | D^\alpha f(x) |^p dx \right)^{1/p} \tag{2.1}$$

We use the standard notations $x = (x_1, \cdots, x_v)$, $D_j = -i\partial/\partial x_j$, $D = (D_1, \cdots, D_v)$

and $D^\alpha = D_1^{\alpha_1} \cdots D_\nu^{\alpha_\nu}$. $\alpha = (\alpha_1, \cdots, \alpha_\nu)$ is a multi-index of non-negative integers, $|\alpha| = \sum_{i=1}^\nu \alpha_i$ and for $\xi \in R^\nu$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_\nu^{\alpha_\nu}$.

Let $L(x,D) = \sum_{|\alpha| \leq \omega} a_\alpha(x) D^\alpha$ be an $m \times n$ differential system defined in $O \subset R^\nu$. For $X \in O$ and $\xi \in R^\nu$, set $\sigma_\omega L(x,\xi) = \sum_{|\alpha| = \omega} a_\alpha(x)\xi^\alpha$.

Let $O' \subset R^{\nu-1}$. Suppose that $L(x, D)$ is defined in a neighborhood in R_+^ν of $\{x; x' \in O'\ x^\nu = 0\}$ and is given by $L(x,D) = \sum_{i=0}^\omega A_i(x', x, D_{x'}) D_{x_\nu}^{\omega - i}$. Let $A_i(0)$ be the differential system in $R^{\nu-1}$ that satisfies $A_i(0) = A_i(x', 0, D_{x'})$. Denote by $\sigma_\omega L(x', \xi', D_t)$ the ordinary differential system, in the variable t, defined by $\sigma_\omega L(x', \xi', D_t) = \sum_{i=0}^\omega \sigma_i(A_i(0))(x', \xi') D_t^{\omega - i}$. Here $R_+^\nu = \{x; x' \in R^{\nu-1}, x_\nu = 0\}$, $D_t = -i\partial/\partial t$ and $\xi^\nu \in R^{\nu-1}$.

Let $\{O_j\}$, $j = 1, \cdots, N$, be a set of local coordinates for $\bar{G}$. Suppose that for $j = 1, \cdots, m$ the points in O_j are coordinates of points in G and that for $j = m+1, \cdots, N$ and $x \in O_j$, $x = (x', x_\nu)$ where $x' \in O_j' \subset R^{\nu-1}$ and x_ν is a "normal" variable. Thus $x_\nu \geqq 0$ and the tuples $(x', 0)$, $x' \in O_j'$, are coordinates of points in ∂G.

Finally let $A(x,D)$ be an $l \times l$ differential system that is infinitely differentiable in $\bar{G}$. Suppose that $A(x,D)$ is elliptic of order ω in $\bar{G}$ (i.e., for $x \in \bar{G}$ and $\xi \neq 0$ the matrix $\sigma_\omega A(x,\xi)$ is non-singular). Let $B_j(x,D)$ $j = 1, \cdots, \omega l/2$ be a $1 \times l$ infinitely differentiable system in $\bar{G}$. Denote by ω_j the order of B_j. We shall need later the following Agmon's conditions for the ray $l_\theta = \{\lambda; \lambda = re^{i\theta},\ r \geqq 0\}$:

Condition I. $\sigma_\omega A(x,\xi)$ has no eigenvalues on l_θ for $x \in \bar{G}$ and $\xi \neq 0$.

Condition II. Let $m + 1 \leqq j \leqq N$. For $x' \in O_j'$, $\xi' \in R^{\nu-1}$ and $\lambda \in l_\theta$ such that $|\xi'| + |\lambda| \neq 0$, the ordinary boundary value problem in x_ν:

$$\sigma_\omega' A(x', \xi', D_{x_\nu}) u = \lambda u \text{ for } x_\nu \geqq 0 \tag{2.2}$$

$$\sigma_{\omega_j}' B_j(x', \xi', D_{x_\nu}) u = g_j \text{ for } x_\nu = 0 \text{ and } j = 1, \cdots, \omega l/2 \tag{2.3}$$

$$\lim_{x_\nu \to \infty} u(x_\nu) = 0 \tag{2.4}$$

has a unique solution for every choice of the complex numbers g_j $j = 1, \cdots, \omega l/2$. See [1] and [5]. Here $A(x, 0)$ is written in terms of the local coordinates O_j.

Observe that if conditions I and II are satisfied on a ray l_θ there exists a $\delta > 0$ such that the same conditions are satisfied on l_ϕ, provided that $|\phi - \theta| \leqq \delta$.

For $f \in C_0^\infty(\bar{G})$, $\tilde{f}$ is the Fourier transform of f given by

$$\tilde{f}(\xi) = \int_{R^\nu} e^{-ix\xi} f(x)\, dx. \tag{2.5}$$

$\tilde{f}_\nu$ is the partial Fourier transform of f defined by

$$\tilde{f}_\nu(\xi', x_\nu) = \int_{R^{\nu-1}} e^{-ix'\xi'} f(x)\, dx' . \tag{2.6}$$

Let $\Gamma(\alpha_1, \alpha_2) = \{\lambda; \alpha_1 \leqq \arg \lambda \leqq \alpha_2\}$ and denote by $\gamma(\alpha_1, \alpha_2)$ the boundary of $\Gamma(\alpha_1, \alpha_2)$ that is positively oriented with respect to $\Gamma(\alpha_1, \alpha_2)$. For $n = 1, 2, \cdots$, $\gamma_n(\alpha_1, \alpha_2) = \gamma(\alpha_1, \alpha_2) \cap \{\lambda; |\lambda| \leqq n\}$ and $\Gamma^\circ(\alpha_1, \alpha_2)$ is the interior of Γ.

Given a Banach space X we denote by $\Sigma(\alpha_1, \alpha_2, c)$ the class of linear operators in X that possess the following properties: A is closed, the domain $D(A)$ of A is dense in X, the spectrum $\sigma(A)$ of A satisfies $\sigma(A) \subset \Gamma(\alpha_1, \alpha_2)$, and there exists a constant c such that for $\lambda \notin \Gamma(\alpha_1, \alpha_2)$ we have

$$\| (\lambda - A)^{-1} \| \leqq c / |\lambda| . \tag{2.7}$$

It is a well known result (cf. [4]) that if $A \in \Sigma(\alpha_1, \alpha_2, c)$ and $\pi/2 < \alpha_1 < \alpha_2 < 3/2\pi$, then A is the infinitesimal generator of an analytic semigroup.

3. Preliminaries

The following Lemma is used in the next section.

LEMMA. 3.1 *Let A be a closed and densely defined linear operator in a Banach space X. Let $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$. Suppose that for $i = 1, 2, \cdots l_{\alpha_i}$ is a ray of minimal growth of the resolvent of A and that $l_{\alpha_i} \subset \rho(A)$. Assume that there exists a constant M such that the estimate*

$$\left\| \frac{1}{2\pi i} \int_\gamma e^{\lambda z} (\lambda - A)^{-1} d\lambda \right\| \leqq M \tag{3.1}$$

holds for $z \in \Gamma^0(\pi/2 - \alpha_1, 3\pi/2 - \alpha_2)$. Here $\gamma = \gamma(\alpha_1, \alpha_2)$. Then the limit

$$\lim_{\substack{z \to 0 \\ z \in \Gamma^0(\pi/2 - \alpha_1,\, 3\pi/2 - \alpha_2)}} \frac{1}{2\pi i} \int_\gamma e^{\lambda z} (\lambda - A)^{-1} x d\lambda \tag{3.2}$$

exists for every $x \in X$. Let E be the bounded operator in X defined by

$$Ex = \lim_{t \to 0} \int_\gamma e^{\lambda t} (\lambda - A)^{-1} x d\lambda. \tag{3.3}$$

Then E is a projection such that for $x \in D(A)$ we have $Ex \in D(A)$ and $AEx = EAx$. The operator AE satisfies $AE \in \Sigma(\alpha_1, \alpha_2, c)$.

PROOF. We shall use the fact that if l_θ is a ray of minimal growth of the resolvent of A, then there exist positive constants ε, c, and R such that for $\lambda \in \Gamma(\theta - \varepsilon, \theta + \varepsilon) \cap \{\lambda; |\lambda| \geqq R\}$ we have $\lambda \in \rho(A)$ and

$$\|(\lambda - A)^{-1}\| \leqq c/|\lambda| . \tag{3.4}$$

To verify this assertion observe that $\mu \in \rho(A)$ and $(\mu-A)^{-1} = ((\mu-\lambda)(\lambda-A)^{-1} + I)^{-1}(\lambda - A)^{-1}$ provided that $\lambda \in \rho(A)$ and that $\|(\mu - \lambda)(\lambda - A)^{-1}\| < 1$. It follows from this remark and from the assumptions of the present Lemma that there exist constants M and $\delta > 0$ such that for $i=1,2$, we have $A \in \Sigma(\alpha_i + \delta - 2\pi, \alpha_i - \delta, c)$.

Let $S(z)$ be the semigroup defined in $\Gamma^0(\pi/2 - \alpha_1, 3\pi/2 - \alpha_2)$ by

$$S(z) = \frac{1}{2\pi i} \int_\gamma e^{\lambda z} (\lambda - A)^{-1} d\lambda . \tag{3.5}$$

To check that $S(z)$ has the semigroup property note that

$$S(z_1)S(z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_\gamma e^{\lambda z}(\lambda - A)^{-1} d\lambda \int_{\gamma'} e^{\mu z}(\mu - A)^{-1} d\mu \tag{3.6}$$

where the curve γ' is obtained from γ by a slight translation to the right. Using 3.6, the resolvent equation $(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}$, and the standard techniques of operational calculus we find that $S(z_1)S(z_2) = S(z_1 + z_2)$.

Since $D(A)$ is dense in X and 3.1 holds, it suffices to show that the limit 3.2 exists for every $x \in D(A)$ in order to prove that this limit exists for every $x \in X$. The relation

$$\lambda(\lambda - A)^{-1} = A(\lambda - A)^{-1} + I \tag{3.7}$$

and the assumption that A is closed imply that for $x \in D(A)$

$$S(z)x = \frac{1}{2\pi i} \int_{\gamma_r} e^{\lambda z}(\lambda - A)^{-1}\lambda^{-1} Ax d\lambda \tag{3.8}$$

where γ_r is the path obtained by joining the two rays on $\gamma \cap \{\lambda; |\lambda| \geqq r\}$ to the arc $\{\lambda; |\lambda| = r, \alpha_1 \leqq \arg\lambda \leqq \alpha_2\}$ and r is chosen so that $\{\lambda; |\lambda| \leqq r\} \subset \rho(A)$. It follows from 3.8 that for $x \in D(A)$, the limit 3.2 exists and that

$$Ex = \frac{1}{2\pi i} \int_{\gamma_r} (\lambda - A)^{-1}\lambda^{-1} Ax d\lambda. \tag{3.9}$$

A consequence of the semigroup property of $S(z)$ is that the bounded operator E defined by 3.3 is a projection. Also, since A is closed, for $x \in D(A)$ and $t>0$, we have $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$. This implies that for $x \in D(A)$, we have $Ex \in D(A)$ and $AEx = EAx$. Note that for $i = 1,2$, we have

(3.10) $$AE \in \Sigma(\alpha_i + \delta - 2\pi,\ \alpha_i - \delta,\ c).$$

This follows from the definition of δ and from the fact that for $\lambda \in \rho(A) - \{0\}$ we have $\lambda \in \rho(AE)$ and

(3.11) $$(\lambda - AE)^{-1} = (\lambda - A)^{-1}E + \lambda^{-1}(I - E).$$

Let $T_1(r)$ be the semigroup defined by $T_1(r) = S(e^{i(\pi/2-\alpha_1+\delta/2)}r) + I - E$ for $r > 0$ and $T_1(0) = I$. It is easily seen that $T_1(r)$ is bounded and strongly continuous for $r \geqq 0$ and it follows from 3.7 that for $r > 0$ we have

(3.12) $$\frac{d}{dr}T_1(r) = e^{i(\pi/2-\alpha_1+\delta/2)}AET_1(r).$$

A consequence of 3.12 is that the infinitesimal generator J_1 of $T_1(r)$ is an extension of $e^{i(\pi/2-\alpha_1+\delta/2)}AE$. Using the Hille-Phillips theorem we find that $\{\lambda; \operatorname{Re}\lambda > 0\} \subset \rho(J_1)$ and that there exists a c such that

(3.13) $$J_1 \in \Sigma(\pi/2 - \delta/3,\ 3\pi/2 + \delta/3, c).$$

The inclusion $\{\lambda; \operatorname{Re}\lambda > 0\} \subset \rho(J_1)$ and 3.10 guarantee that

$$\rho(J_1) \cap \rho(e^{i(\pi/2-\alpha_1+\delta/2)}AE) \neq \emptyset$$

and this implies that $J_1 = e^{i(\pi/2-\alpha_1+\delta/2)}AE$. Combining 3.13 with 3.10 we find that

(3.14) $$AE \in \Sigma(\alpha_1,\ \alpha_1 + \pi, c).$$

Similarly $e^{i(3\pi/2-\alpha_2-\delta/2)}AE = J_2$ where J_2 is the infinitesimal generator of the semigroup $T_2(r)$ given by $T_2(r) = S(e^{i(3\pi/2-\alpha_2-\delta/2)}r) + I - E$ for $r > 0$ and $T_2(0) = I$, and we have

(3.15) $$AE \in \Sigma(\alpha_2 - \pi,\ \alpha_2,\ c).$$

A consequence of 3.14 and 3.15 is that $A \in \Sigma(\alpha_1, \alpha_2, c)$.

Let G be a bounded domain in R^ν with an infinitely differentiable boundary ∂G. Let $A(x,D)$ be an $l \times l$ elliptic system of even order ω that is infinitely differentiable in $\bar{G}$. Let $B_j(x,D)$ $j = 1,\cdots,\omega l/2$ be the boundary operators that are associated with the elliptic system. Suppose that the $1 \times l$ differential system $B_j(x,D)$ has order $\omega_j < \omega$ and that $B_j(x,D)$ is infinitely differentiable in $\bar{G}$. Suppose that $1 < p < \infty$ and denote by $H^{\omega,p}(G,B)$ the closure, in $H^{\omega,p}(G)$, of the set of functions u in $C^\infty(\bar{G})$ that satisfy $B_ju = 0$ on ∂G for $j = 1,\cdots\omega l/2$. As usual we associate with $A(x,D)$, $B_j(x,D)$ $j = 1,\cdots\omega l/2$ and G the unbounded operator A_B^p in $H^{0,p}(G)$ that satisfies $D(A_B^p) = H^{\omega,p}(G,B)$ and $A_B^pu = A(x,D)u$ for $u \in H^{\omega,p}(G,B)$.

The question of existence of rays of minimal growth of the resolvent of A_B^p was investigated in [1], [5] and [6]. In the remainder of this section, we cite for convenience some known results of these works that will be used later on.

Agmon's conditions I and II of Section 1 guarantee that the ray l_θ is a ray of minimal growth of the resolvent of A_B^p([1], [5] and [6]). Also if conditions I and II are satisfied for $\theta \in [\alpha_1, \alpha_2]$, there exist constants c and R such that $\lambda \in \rho(A_B^p)$ and

$$\| (\lambda - A_B^p)^{-1} \| \leqq c / |\lambda| . \tag{3.16}$$

for $\lambda \in \Gamma(\alpha_1, \alpha_2)$ with $|\lambda| \geqq R$.

We shall use in the sequel a parametrix for $(\lambda - A_B^p)^{-1}$ with $\lambda \in l_\theta$ that is defined in [5] provided that conditions I and II are satisfied on l_θ. The parametrix $P_0(\lambda)$ of order zero is given by

$$P_0(\lambda) = \sum_{j=1}^{N} C_j(\lambda) - \sum_{j=m+1}^{N} D_j(\lambda) \tag{3.17}$$

where $C_j(\lambda)$ $(\lambda \in l_\theta)$ is a pseudodifferential operator and for $f \in C_0^\infty(G)$

$$C_j(\lambda) f(x) = (2\pi)^{-\nu} \psi_j(x) \int e^{ix\xi} \theta(\xi, \lambda) (\sigma_\omega A(x\xi) - \lambda)^{-1} (\widetilde{\phi f})(\xi) d\xi. \tag{3.18}$$

$D_j(\lambda)$ is defined for $f \in C_0^\infty(G)$ by

$$D_j(\lambda) f(x) = (2\pi)^{-\nu+1} \psi_j(x) \int e^{ix'\xi'} \theta'(\xi', \lambda) \tilde{d}(x', x_\nu, \xi', s, \lambda) (\widetilde{\phi_j f})_\nu(\xi', s) d\xi' ds \tag{3.19}$$

where ϕ_j, $j = 1, \cdots, N$ is a partition of unity of $\bar{G}$ subordinate to the system of coordinate neighborhoods O_j with the properties described in Section 1. The scalar function ψ_j is infinitely differentiable in $\bar{G}$ and the support of ψ_j is contained in O_j. $\theta(\xi, \lambda)$ is infinitely differentiable for $\xi \in R^\nu$ and $\lambda \in l_\theta$, $\theta(\xi, \lambda) = 1$ for $|\xi|^2 + |\lambda|^{2/\omega} \geqq 1$ and $\theta(\xi, \lambda) = 0$ for $|\xi|^2 + |\lambda|^{2/\omega} \leqq \frac{1}{2}$. Similarly $\theta'(\xi', \lambda)$ is infinitely differentiable for $\xi' \in R^{\nu-1}$ and $\lambda \in l_\theta$, $\theta'(\xi', \lambda) = 1$ for $|\xi'|^2 + |\lambda|^{2/\omega} \geqq 1$ and $\theta'(\xi', \lambda) = 0$ for $|\xi'|^2 + |\lambda|^{2/\omega} \leqq \frac{1}{2}$.

The $l \times l$ matrix $\tilde{d}$ is defined for $x_\nu \geqq 0$, $s \geqq 0$ and for ξ' and $\lambda \in l_\theta$ such that $\xi'| + |\lambda| \neq 0$ and satisfies the estimates

$$\begin{aligned} &| D_{x'}^{\delta'} D_{\xi'}^{\varepsilon'} D_\lambda^p d(x', x_\nu, \varepsilon', s, \lambda) | \\ &\leqq c_{\delta'\varepsilon'p} \exp[-c_1(x_\nu + s)(|\xi'| + |\lambda|^{1/\omega})] (|\xi'| + |\lambda|^{1/\omega})^{1-\omega-|\varepsilon'|-\omega p} \end{aligned} \tag{3.20}$$

See [5]. The following Lemmas of [6] are used in Section [4]. These Lemmas are proved in [6] and are used there to estimate the L_p norms of singular integral operators that are derived from $C_j(\lambda)$ and $D_j(\lambda)$.

LEMMA 1. *Let $1 < p < \infty$, $0 < R < \infty$. There is a constant $c = c(p, \nu, R)$ such that if $k(x,\xi)$ vanishes for $|x| \geqq R$ and $|\xi|^{|\beta|}|D_x^\alpha D_\xi^\beta k(x,\xi)| \leqq 1$ for $|\alpha| \leqq \nu + 1$ and $|\beta| \leqq \nu$ then the estimate*

$$(3.21)\qquad \left|(2\pi)^{-\nu}\int e^{ix\xi}k(x,\xi)\tilde{f}(\xi)d\xi\right|_{L_p(R^\nu)} \leqq c|f|_{L_p(R^\nu)}$$

holds for $f \in c_0^\infty(R^\nu)$.

Given k and f that satisfy the assumptions of Lemma 1 we write

$$(3.22)\qquad Op(k)f(x) = (2\pi)^{-\nu}\int e^{ix\xi}k(x,\xi)\tilde{f}(\xi)\,d\xi.$$

LEMMA 2. *Let $1 < p < \infty$. There is a constant $c = c(p,\nu,R)$ such that if $k(x', x_\nu, \xi', \lambda)$ has support in $|x'| \leqq R$ and satisfies*

$$(3.23)\qquad |\xi'|^{|\beta'|}|D_{x'}^{\alpha'}D_{\xi'}^{\beta'}k(x',x_\nu,\xi',s)| \leqq (x_\nu + s)^{-1}$$

for $s > 0$, $x_\nu > 0$, $|\beta'| \leqq \nu$ and $|\alpha'| \leqq \nu$ then the estimate

$$(3.24)\qquad \left|(2\pi)^{-\nu+1}\int e^{ix'\xi'}k(x',x_\nu,\xi',s)\tilde{f}_\nu(\xi',s)\,d\xi' ds\right|_{L_p(R_+^\nu)} \leqq c|f|_{L_p(R^\nu)}$$

holds for every $f \in c_0^\infty(R^\nu)$ with support in the interior of R_+^ν. As usual $R_+^\nu = \{x, x \in R^\nu, x_\nu \geqq 0\}$.

Given k and f that satisfy the assumptions of this Lemma we write

$$(3.25)\qquad O'p(k)f(x) = (2\pi)^{-\nu+1}\int e^{ix'\xi'}k(x',x_\nu,\xi',s)\tilde{f}_\nu(\xi',s)d\xi' ds.$$

LEMMA 3. *Let $1 < p < \infty$. Suppose that $k(x',x_\nu,\xi',s,r)$ vanishes for $|x'| \geqq R$ and that there is a constant c such that*

$$(3.26)\qquad |D_{x'}^{\alpha'}D_{\xi'}^{\beta'}k(x',x_\nu,\xi',s,r)| \leqq e^{-c(x_\nu+s)(|\xi'|+r)}(|\xi'|+r)^{1-\omega-|\beta'|}$$

for $|\alpha'| \leqq \nu$ and $|\beta'| \leqq \nu$ and with $\omega \geqq 1$. Then for each p there is a constant $c_1 = c_1(p,c,R,\nu)$ such that for $f \in c_0^\infty(R^\nu)$ with support in the interior of R_+^ν we have

$$(3.27)\qquad \left|O'p\left(\int_r^{r_2} r^{\omega-1}\,k dr\right)f\right|_{L^p(R_+^\nu)} \leqq c_1|f|_{L_p(R_+^\nu)}$$

for all $r_2 \geqq r_1 \geqq 1$.

Finally we observe that the following estimates are proved in [5] and [6]:

Suppose that conditions I and II are satisfied on l_θ and choose R so that $\lambda \in \rho(A)$ for $\lambda \in l_\theta$ with $|\lambda| \geqq R$. Then there exists a constant c such that for $\lambda \in l_\theta$ and $f \in C_0^\infty(G)$ we have

$$\|C_i(\lambda)f\| \leqq c(1+|\lambda|)^{-1}\|f\| \qquad i = 1,\cdots,N \tag{3.28}$$

and

$$\|D_i(\lambda)f\| \leqq c(1+|\lambda|)^{-1}\|f\| \qquad i = m+1,\cdots,N \tag{3.29}$$

and there exists a constant c such that

$$\|((\lambda - A_B^p)^{-1} - P_0(\lambda))f\| \leqq c(1+|\lambda|^{1+1/\omega})^{-1}\|f\| \tag{3.30}$$

provided that $\lambda \in l_\theta$, $|\lambda| \geqq R$ and $f \in C_0^\infty(G)$. The norms mentioned in estimates (2.28)–(2.30) are the norms of elements in $H^{0,p}(G)$.

For $\lambda \in l_\theta$ we continue to denote by $C_i(\lambda)$, $D_i(\lambda)$ and $P_0(\lambda)$ the extensions of $C_i(\lambda)$, $D_i(\lambda)$ and $P_0(\lambda)$ to bounded operators in $H^{0,p}(G)$.

4. Theorems and proofs

THEOREM 4.1. *Let $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$. Suppose that conditions I and II are satisfied on l_{α_i} and that $l_{\alpha_i} \subset \rho(A_B^p)$ for $i = 1,2$. Then there exists a constant M such that for $z \in \Gamma^0(\pi/2 - \alpha_1, 3\pi/2 - \alpha_2)$, we have*

$$\left\|\int_\gamma e^{\lambda z}(\lambda - A_B^p)^{-1}\right\| \leqq M. \tag{4.1}$$

PROOF. The proof of Theorem 4.1 depends on the following Lemmas.

LEMMA 4.1 *Assume that conditions I and II are satisfied on l_{α_i} and that $l_{\alpha_i} \subset \rho(A_B^p)$ for $i = 1,2$. Suppose that $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$. Then there exists a constant M such that*

$$\left\|\frac{1}{2\pi i}\int_\gamma (e^{\lambda z}(\lambda - A_B^p)^{-1} - P_0(\lambda))d\lambda\right\| \leqq M \tag{4.2}$$

for $z \in \Gamma^0(\pi/2 - \alpha_1, 3\pi/2 - \alpha_2)$.

PROOF. Lemma 4.1 is an immediate consequence of estimate (3.30).

LEMMA 4.2. *Assume that conditions I and II are satisfied on l_{α_i} for $i = 1,2$ and suppose that $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$. Set $C(\lambda) = C_j(\lambda)$ for $\lambda \in \gamma$ and for some j with $1 \leqq j \leqq N$. Then there exists a constant M such that*

$$\left\|\frac{1}{2\pi i}\int_\gamma e^{\lambda z}C(\lambda)d\lambda\right\| \leqq M \tag{4.3}$$

for $z \in \Gamma^0(\pi/2-\alpha_1, 3\pi/2-\alpha_2)$.

PROOF. Let χ be a real valued infinitely differentiable function such that $\chi(\xi)=0$ for $|\xi| \leqq 1$ and $\chi(\xi)=1$ for $|\xi| \geqq 2$. For $z \in \Gamma^0(\pi/2-\alpha_1, 3\pi/2-\alpha_2)$ set

$$a_1(x,\xi,z) = \frac{1}{2\pi i}\psi(x)\int_\gamma e^{\lambda z}\chi(\xi)(\sigma_\omega A(x,\xi)-\lambda)^{-1}d\lambda \tag{4.4}$$

and

$$a_2(x,\xi,z) = \frac{1}{2\pi i}\psi(x)\int_\gamma e^{\lambda z}\theta(\xi,\lambda)(1-\chi(\xi))(\sigma_\omega A(x,\xi)-\lambda)^{-1}d\lambda \tag{4.5}$$

Choose non negative numbers r_1 and r_2 so that for every x and $\xi \neq 0$ the eigenvalues of $\sigma_\omega A(x,\xi)$ satisfy $2r_1|\xi| \leqq |\lambda|^{1/\omega} \leqq \frac{1}{2} r_2|\xi|$. Let $L^-(R)$ be the boundary of $\Gamma(\alpha_1,\alpha_2) \cap \{\lambda; r_1 R \leqq |\lambda|^{1/\omega} \leqq r_2 R\}$. Let $z \in \Gamma^0(\pi/2-\alpha_1, 3\pi/2-\alpha_2)$ and suppose that $|\alpha| \leqq \nu+1$ and $|\beta| \leqq \nu$. Then $|\xi|^\beta D_x^\alpha D_\xi^\beta a_1(x,\xi,z)$ is a linear combination of terms of the type

$$\frac{1}{2\pi i}|\xi|^\beta(D_x^{\alpha_1}\psi(x))\int_{L^-(|\xi|)} e^{\lambda z}(D_\xi^{\beta_1}\chi(\xi))D_x^{\alpha_2}D_\xi^{\beta_2}(\sigma_\omega A(x,\xi)-\lambda)^{-1}d\lambda \tag{4.6}$$

with $|\alpha_1|+|\alpha_2| = |\alpha|$ and $|\beta_1|+|\beta_2| = |\beta|$. Setting $\lambda = |\xi|^\omega\mu$ it is easily seen that if $|\beta_1| = 0$ there exists a c such that (4.6) is bounded by

$$c \sup_{x,|\eta|=1}\int_{L^-(1)} |D_x^{\alpha_1}\psi(x)|\,|D_x^{\alpha_2}D_\eta^{\beta_2}(\sigma_\omega A(x,\eta)-\mu)^{-1}|d\mu. \tag{4.7}$$

For $|\beta_1| \neq 0$, $D_\xi^{\beta_1}\chi(\xi)$ vanishes for $|\xi| \leqq 1$ and for $|\xi| \geqq 2$. For $1 \leqq |\xi| \leqq 2$ the path of integration in (4.6) can be deformed to the boundary of $\Gamma(\alpha_1,\alpha_2) \cap \{\lambda;\ r_1 \leqq |\lambda|^{1/\omega} \leqq 2r_2\}$ and the integral thus obtained is bounded by a constant independent of x, ξ and z. As a result there exists a c such that

$$|\xi^\beta|\,|D_x^\alpha D_\xi^\beta a_1(x,\xi,z)| \leqq c\,. \tag{4.8}$$

Now $a_2(x,\xi,z) = 0$ for $|\xi| \geqq 2$. For $|\xi| \leqq 2$, the path of integration in (4.5) can be deformed to the boundary of $\Gamma(\alpha_1,\alpha_2) \cap \{\lambda; |\lambda|^{1/\omega} \leqq 2r_2\}$. It follows from the definition of $\theta(\xi,\lambda)$ that there exists a c such that

$$|\xi|^\beta|D_x^\alpha D_\xi^\beta a_2(x,\xi,z)| \leqq c \tag{4.9}$$

for $|\alpha| \leqq \nu+1$, $|\beta| \leqq \nu$ and $z \in \Gamma^0(\pi/2-\alpha_1, 3\pi/2-\alpha_2)$.

Using (4.4), (4.5) and the definition of $C(\lambda)$ we find that for $f \in C_0^\infty(G)$

$$\left(\left(\frac{1}{2\pi i}\int_\gamma e^{\lambda z}C(\lambda)d\lambda\right)f\right)(x) = (2\pi)^{-\nu}\int e^{ix\xi}(a_1(x,\xi,z)+a_2(x,\xi,z))\hat{f}(\xi)d\xi. \tag{4.10}$$

(4.10) and estimates (4.8) and (4.9) combined with Lemma 1 guarantee the existence of a constant M such that (4.3) is satisfied.

LEMMA 4.3. *Suppose that conditions I and II are satisfied on the ray* l_θ. *Let* $D(\lambda)$, $\lambda \in l_\theta$, *be one of the families* $D_j(\lambda)$ *given by* (3.19). *Then there exists a constant M such that*

$$\left|\frac{1}{2\pi i}\int_{l_\theta} e^{\lambda z} D(\lambda)d\lambda\right| \leqq M \tag{4.11}$$

for all z such that $\mathrm{Re}(ze^{i\theta}) < 0$.

PROOF. It follows from estimate (3.20) and from the definition of $D(\lambda)$ that there exist c_1 and c_2 such that for $\lambda \in l_\theta$ and z with $\mathrm{Re}(ze^{i\theta}) < 0$, for $|\alpha'| \leqq \nu$ and for $|\beta'| \leqq \nu$ the estimate

$$\begin{aligned} &\left| D_{x'}^{\alpha'} D_{\xi'}^{\beta'} e^{\lambda z}\psi(x)\theta'(\xi',\lambda)\tilde{d}(x',x_\nu,\xi',s,\lambda)\right| \\ &\qquad \leqq c_1 e^{-c_2(x_\nu+s)(|\xi'|+/|\lambda|^{1/\omega})}(|\xi'|+|\lambda|^{1/\omega})^{1-\omega-|\beta'|} \end{aligned} \tag{4.12}$$

is satisfied. It follows from (4.12) and Lemma 3 of Section 3 that (4.11) holds with l_θ replaced by $l_\theta \cap \{\lambda; |\lambda| \leqq n\}$ and with a constant M independent of $n = 1,2,\cdots$. A consequence of (3.29) is that for z with $\mathrm{Re}(ze^{i\theta}) < 0$

$$\frac{1}{2\pi i}\int_{l_\theta} e^{\lambda z}D(\lambda)d\lambda = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{l_\theta\cap\{\lambda;|\lambda|\leqq n\}} e^{\lambda z}D(\lambda)d\lambda . \tag{4.13}$$

Hence (4.11) follows.

A consequence of Theorem 4.1 is the following Theorem.

THEOREM 4.2. *Assume that conditions I and II are satisfied on* l_{θ_i} *and that* $l_{\theta_i} \subset \rho(A_B^p)$ *for* $i = 1,2$. *Let* $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$ *and suppose that* $\theta_2 - \theta_1 \leqq \pi$. *For* $f \in D(A_B^p)$ *let*

$$E_p f = \frac{1}{2\pi i}\int_{\gamma_r(\theta_1,\theta_2)} \lambda^{-1}(\lambda - A_B^p)^{-1}A_B^p f d\lambda . \tag{4.14}$$

Here r is chosen so that $\{\lambda; |\lambda| \leqq r\} \subset \rho(A_B^p)$. *Then* E_p *has an extension to a bounded projection in* $H^{0,p}(G)$ *that we continue to denote by* E_p. *For* $f \in D(A_B^p)$ *we have* $E_p f \in D(A_B^p)$ *and* $A_B^p E_p f = E_p A_B^p f$. *Furthermore* $\sigma(A_B^p E_p) - \{0\} = \sigma(A_B^p) \cap \Gamma(\theta_1,\theta_2)$ and $A_B^p E_p \in \Sigma(\theta_1,\theta_2,c)$.

PROOF. The assumptions of this theorem imply that conditions I and II for the elliptic system $e^{i\theta}A(x,D)$ and the boundary operators $B_j(x,D)$ $j = 1,\cdots \omega l/2$, are satisfied on the rays $l_{\theta\theta_1}$ and $l_{\theta\theta_2}$. Also $e^{i\theta}A_B^p$ is the operator in $H^{0,p}(G)$ that is associated with $e^{i\theta}A(x,D)$ and $B_j(x,D)$ $j = 1\cdots\omega l/2$. It follows from these remarks that we may assume without loss of generality that $\pi/2 \leqq \theta_1 < \theta_2 \leqq 3\pi/2$. As has already been mentioned before there exists a $\delta > 0$ such that conditions I and II are satisfied on l_θ for $\theta \in [\theta_1, \theta_1 + 2\delta] \cup [\theta_2, \theta_2 - 2\delta]$ and such that $A_B^p \in \Sigma(\theta_1 + 2\delta - 2\pi, \theta_1, c)$ and $A_B^p \in \Sigma(\theta_2 - 2\pi, \theta_2 - 2\delta, c)$. This implies that $A_B{}^p$ satisfies the assumptions of Theorem 4.1 with $\alpha_1 = \theta_1 + \delta$ and $\alpha_2 = \theta_2 - \delta$. Also it is easy to verify that for $f \in D(A_B^p)$

$$E_p f = \frac{1}{2\pi i} \int_{\gamma_r(\alpha_1, \alpha_2)} \lambda^{-1}(\lambda - A_B^p)\, A_B^p f d\lambda\,. \tag{4.15}$$

Using Theorem 4.1, Lemma 3.1 and observing that 3.9 holds, we find that E_p has an extension to a bounded projection in $H^{0,p}(G)$. Also it follows from Lemma 3.1 and from the definition of δ that to complete the proof it is sufficient to check that $\sigma(A_B^p E_p) \supset \sigma(A_B^p) \cap \Gamma(\theta_1, \theta_2)$. Suppose that $\lambda_0 \in \sigma(A_B^p) \cap \Gamma(\theta_1, \theta_2)$ and let f be an eigenfunction of A_B^p corresponding to the eigenvalue λ_0. Then as is easily checked

$$\frac{1}{2\pi i} \int_{\gamma_r(\theta_1, \theta_2)} \lambda^{-1}(\lambda - A_B^p)^{-1} A_B^p f d\lambda = f. \tag{4.16}$$

A consequence of (4.15) and (4.16) is that $E_p f = f$, $A_B^p E_p f = \lambda_0 f$ and $\lambda_0 \in \sigma(A_B^p E_p)$

It will be convenient in the sequel to denote the operator E_p of Theorem 4.2 by $E(\theta_1, \theta_2)$.

THEOREM 4.3. *Assume that* $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$, $-\pi/2 < \phi_1 < \phi_2 < \pi/2$ *and that conditions I and II are satisfied on* l_θ *provided that* $\theta \notin (\phi_1, \phi_2) \cup (\theta_1, \theta_2)$. *Suppose that for* $i = 1, 2$, *we have* $l_{\theta_i} \in \rho(A_B^p)$ *and* $l_{\phi_i} \in \rho(A_B^p)$. *Let* $E_1 = E(\theta_1, \theta_2)$ *and let* $E_2 = E(\phi_1, \phi_2)$. *Denote by* E_0 *the spectral projection of* A_B^p *associated with the set* $\{\lambda;\, \lambda \notin \Gamma(\theta_1, \theta_2) \cup \Gamma(\phi_1, \phi_2)\}$. *Then*

$$H^{0,p}(G) = \sum_{i=0}^{2} \oplus E_i H^{0,p}(G). \tag{4.17}$$

The limits

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\gamma^n(\theta_1, \theta_2)} (\lambda - A_B^p)^{-1} f d\lambda \tag{4.18}$$

and

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\gamma^n(\phi_1, \phi_2)} (\lambda - A_B^p)^{-1} f d\lambda \tag{4.19}$$

exist for every $f \in H^{0,p}(G)$. *Also*

(4.20) $$E_1 f = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)} (\lambda - A_B^p)^{-1} f d\lambda + (\theta_2 - \theta_1)/2\pi f$$

and

(4.21) $$E_2 f = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\gamma^n(\phi_1,\phi_2)} (\lambda - A_B^p)^{-1} f d\lambda + (\phi_2 - \phi_1)/2\pi f.$$

PROOF. For $t > 0$ let

(4.22) $$S_1(t) = \frac{1}{2\pi i}\int_{\gamma(\phi_1,\phi_2)} e^{\lambda t}(\lambda - A_B^p)^{-1} d\lambda$$

and put

(4.23) $$S_2(t) = \frac{1}{2\pi i}\int_{\gamma(\phi_1,\phi_2)} e^{-\lambda t}(\lambda - A_B^p)^{-1} d\lambda.$$

Then for $f \in H^{0,p}(G)$ and $i = 1,2$, $E_i f = \lim_{t\to 0} S_i(t) f$. Also

(4.24) $$E_0 = \frac{1}{2\pi i}\int_\partial (\lambda - A_B^p)^{-1} d\lambda$$

where ∂ is the boundary of $\{\lambda;\, \lambda \notin \Gamma(\theta_1,\theta_2) \cup \Gamma(\phi_1,\phi_2),\, |\lambda| \leqq R\}$ and R is chosen so that $\{\lambda;\, \lambda \notin (\theta_1,\theta_2) \cup \Gamma(\phi_1,\phi_2),\, |\lambda| \geqq R\} \subset \rho(A_B^p)$.

It is easy to verify, using the standard techniques of operational calculus, that for $t > 0$ $S_1(t)S_2(t) = S_2(t)S_1(t) = 0$ and that for $i = 1,2$, $S_i(t)E_0 = E_0 S_i(t) = 0$. As a result $E_iE_j = E_jE_i = 0$ provided that $i,j = 1,2,3$ and $i \neq j$, and this implies that the sum $\sum_{i=0}^2 \oplus E_i H^{0,p}(G)$ is direct. To check that $\sum_{i=0}^2 \oplus E_i H^{0,p}(G) = H^{0,p}(G)$ it is sufficient to verify that for $f \in D(A_B^p)$ we have $\sum_{i=0}^2 E_i f = f$. Let $f \in D(A_B^p)$. Using (3.9) we find that

(4.25)
$$E_1 f + E_2 f = \frac{1}{2\pi i}\int_{\gamma_r(\theta_1,\theta_2)} \lambda^{-1}(\lambda - A_B^p)^{-1} A_B^p f d\lambda + \frac{1}{2\pi i}\int_{\gamma_r(\phi_1,\phi_2)} \lambda^{-1}(\lambda - A_B^p)^{-1} A_B^p f d\lambda$$

where r is chosen so that $\{\lambda;\, |\lambda| \leqq r\} \subset \rho(A_B^p)$. Observing that there exists a constant c such that

(4.26) $$\|(\lambda - A_B^p)^{-1}\| \leqq c/|\lambda|$$

provided that $\lambda \notin \Gamma(\theta_1,\theta_2) \cup \Gamma(\phi_1,\phi_2)$ and $|\lambda| \geqq R$ and using (3.7) we find that

(4.27) $$E_1 f + E_2 f = -\frac{1}{2\pi i}\int_\partial (\lambda - A_B^p)^{-1} f d\lambda + f$$

i.e., $E_1 f + E_2 f = -E_0 f + f$.

Furthermore for $f\in D(A_B^p)$ we have

$$\frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)}(\lambda-A_B^p)^{-1}fd\lambda=\frac{1}{2\pi i}\int_{\gamma_r^n(\theta_1,\theta_2)}\lambda^{-1}(\lambda-A_B^p)^{-1}A_B^pfd\lambda+\frac{1}{2\pi i}\int_{\gamma_r^n(\theta_1,\theta_2)}\lambda^{-1}fd\lambda \tag{4.28}$$

and the limit 4.18 exists. Here $\gamma_r^n(\theta_1\theta_2)=\gamma_r(\theta_1\theta_2)\cap\{\lambda;\,|\lambda|\leqq n\}$.

Using (3.11) and the fact that $A_BE_1^p\in\Sigma(\theta_1,\theta_2,c)$, we verify that there exists a constant c such that

$$\left\|\frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)}(\lambda-A_B^p)^{-1}E_1d\lambda\right\|\leqq c \tag{4.29}$$

for $n=1,2,\cdots$. Similarly there exists a constant c such that (4.29) holds with E_1 replaced by E_2. The boundedness of AE_0 and (3.11) guarantee that there exists a constant c such that (4.29) holds with E_1 replaced by E_0. It follows from (4.17) that there exists a c such that

$$\left\|\frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)}(\lambda-A_B^p)^{-1}\,d\lambda\right\|\leqq c \tag{4.30}$$

for $n=1,2,\cdots$. The relation (4.30) combined with the existence of the limit (4.18) for $f\in D(A_B^p)$ guarantee that the limit (4.18) exists for every $f\in H^{0,p}(G)$.

The existence of the limit (4.19) is proved in a similar way.

To show that (4.20) holds it is sufficient to check that (4.20) is satisfied for $f\in D(A_B^p)$. For such f, (4.20) follows from (3.9) and from the relation

$$\frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)}\lambda^{-1}(\lambda-A_B^p)^{-1}A_B^pfd\lambda=\frac{1}{2\pi i}\int_{\gamma^n(\theta_1,\theta_2)}(\lambda-A_B^p)^{-1}fd\lambda-\frac{1}{2\pi i}\int_{\gamma_r^n(\theta_1,\theta_2)}\lambda^{-1}fd\lambda. \tag{4.31}$$

The proof of 4.21 is similar.

From Theorem 4.2 and 4.3 we get

COROLLARY 4.1. *Suppose that conditions I and II are satisfied on* $l_{\pi/2}$ *and* $l_{-\pi/2}$. Then there exist bounded projections E_p^- and E_p^+ in $H^{0,p}(G)$ such that the following is satisfied: $H^{0,p}(G)=E_p^-H^{0,p}(G)\oplus E_p^+H^{0,p}(G)$. A_B^p is completely reduced by this direct sum decomposition and each of the operators $A_B^pE_p^-$ and $-A_B^pE_p$ is the infinitesimal generator of an analytic semigroup.

PROOF. Let $\lambda_0 \in \rho(A_B^p)$. Replacing A_B^p by $A_B^p - \lambda_0 I$ if $0 \notin \sigma(A_B^p)$ we conclude that it is sufficient to prove the corollary with the additional assumption $0 \in \rho(A_B^p)$. It follows from the assumptions on $l_{\pm\pi/2}$ that there exists a $\delta > 0$ such that conditions I and II are satisfied on l_θ provided that $|\theta \pm \pi/2| \leqq \delta$. The discreteness of the spectrum of A_B^p and the assumption that $0 \in \rho(A_B^p)$ guarantee that the assumptions of Theorem 4.3 are satisfied for some θ_i and ϕ_i $i = 1,2$. Put $E_p^- = E_1 + E_0$ and let $E_p^+ = E_2$ where E_i, $i = 0,1,2$, are the projections defined in Theorem 4.3. A consequence of Theorem 4.3 is that

$$H^{0,p}(G) = E_p^- H^{0,p}(G) \oplus E_p^+ H^{0,p}(G)$$

and that A_B^p is completely reduced by this direct sum decomposition. Theorem 4.2 implies that $A_B^p E_1 \in \Sigma(\theta_1, \theta_2, c)$ and that $A_B^p E_2 \in \Sigma(\phi_1, \phi_2, c)$. It follows from these relations that each of the operators $A_B^p E_1$ and $-A_B^p E_2$ is the infinitesimal generator of an analytic semigroup (cf. [4]). Also, since $A_B^p E_0$ is bounded, the same result holds for $A_B^p E_1 + A_B^p E_0$.

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